

DIAGONAL RECURRENCE RELATIONS, INEQUALITIES, AND MONOTONICITY RELATED TO STIRLING NUMBERS

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ABSTRACT. In the paper, the author derives several “diagonal” recurrence relations, constructs some inequalities, finds monotonicity, and poses a conjecture related to Stirling numbers of the second kind.

1. INTRODUCTION

In mathematics, Stirling numbers arise in a variety of combinatorics problems. They are introduced in the eighteen century by James Stirling. There are two kinds of Stirling numbers: Stirling numbers of the first and second kinds. Some properties and recurrence relations of Stirling numbers of these two kinds are collected in, for example, [1, Chapter V].

Some Stirling number of the second kind $S(n, k)$ is the number of ways of partitioning a set of n elements into k nonempty subsets. It may be computed by

$$S(n, k) = \frac{1}{k!} \sum_{i=0}^k (-1)^i \binom{k}{i} (k-i)^n \quad (1.1)$$

and may be generated by

$$\frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n, k) \frac{x^n}{n!}, \quad k \in \{0\} \cup \mathbb{N}. \quad (1.2)$$

In this paper, we will derive several “diagonal” recurrence relations, construct some inequalities, and find monotonicity related to $S(n, k)$. By the way, we will also pose a conjecture on monotonicity and logarithmic concavity of sequences related to Stirling numbers of the second kind $S(n, k)$.

2. SEVERAL “DIAGONAL” RECURRENCE RELATIONS OF $S(n, k)$

In [1, p. 209], two “vertical” and two “horizontal” recurrence relations for $S(n, k)$ were listed. Relative to the words “vertical” and “horizontal”, we may call the following formulas (2.1) and (2.2) the “diagonal” recurrence relations for Stirling numbers of the second kind $S(n, k)$.

Theorem 2.1. *For $n > k \geq 0$, we have*

$$S(n, k) = \binom{n}{k} \sum_{\ell=1}^{n-k} (-1)^\ell \frac{\binom{k}{\ell}}{\binom{n-k+\ell}{n-k}} \sum_{i=0}^{\ell} (-1)^i \binom{n-k+\ell}{\ell-i} S(n-k+i, i) \quad (2.1)$$

2010 *Mathematics Subject Classification.* Primary 11B73; Secondary 05A18, 11B83, 11R33, 26D15, 33B10, 44A10.

Key words and phrases. Stirling number of the second kind; diagonal recurrence relation; inequality; logarithmically convex sequence; monotonicity; Faà di Bruno formula; Bell polynomials of the second kind; conjecture.

This paper was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{L}\mathcal{A}\mathcal{T}\mathcal{E}\mathcal{X}$.

$$= (-1)^n \sum_{i=2k-n}^{k-1} (-1)^i \binom{n}{i} \binom{i-1}{2k-n-1} S(n-i, k-i), \quad (2.2)$$

where the conventions that

$$\binom{0}{0} = 1, \quad \binom{-1}{-1} = 1, \quad \text{and} \quad \binom{p}{q} = 0 \quad (2.3)$$

for $p \geq 0 > q$ are adopted.

Proof. The equation (1.2) may be rearranged as

$$\left(\frac{e^x - 1}{x} \right)^k = \sum_{n=0}^{\infty} \frac{S(n+k, k)}{\binom{n+k}{k}} \frac{x^n}{n!}, \quad k \in \{0\} \cup \mathbb{N}. \quad (2.4)$$

Consequently, as coefficients of the power series expansion of the function $\left(\frac{e^x - 1}{x} \right)^k$,

$$\frac{S(n+k, k)}{\binom{n+k}{k}} = \lim_{x \rightarrow 0} \frac{d^n}{d x^n} \left[\left(\frac{e^x - 1}{x} \right)^k \right],$$

that is, for $n \geq k \geq 0$,

$$S(n, k) = \binom{n}{k} \lim_{x \rightarrow 0} \frac{d^{n-k}}{d x^{n-k}} \left[\left(\int_1^e u^{x-1} d u \right)^k \right]. \quad (2.5)$$

In combinatorics, Bell polynomials of the second kind, or say, the partial Bell polynomials, denoted by $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$, may be defined by

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \leq i \leq n, \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^n i \ell_i = n \\ \sum_{i=1}^n \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!} \right)^{\ell_i} \quad (2.6)$$

for $n \geq k \geq 0$, and the well-known Faà di Bruno formula may be described in terms of Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ by

$$\frac{d^n}{d x^n} f \circ g(x) = \sum_{k=1}^n f^{(k)}(g(x)) B_{n,k}(g'(x), g''(x), \dots, g^{(n-k+1)}(x)). \quad (2.7)$$

See [1, p. 134, Theorem A] and [1, p. 139, Theorem C]. In [5, Theorem 1] and [21, Example 4.2], it was derived that

$$B_{n,k} \left(\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n-k+2} \right) = \frac{n!}{(n+k)!} \sum_{i=0}^k (-1)^{k-i} \binom{n+k}{k-i} S(n+i, i). \quad (2.8)$$

See also [7] and closely related references therein. Consequently, we may obtain the following conclusions:

(1) When $1 \leq m \leq k$,

$$\frac{d^m}{d x^m} \left[\left(\int_1^e u^{x-1} d u \right)^k \right] = \sum_{\ell=1}^m \frac{k!}{(k-\ell)!} \left(\int_1^e u^{x-1} d u \right)^{k-\ell}$$

$$\begin{aligned}
 & \times B_{m,\ell} \left(\int_1^e u^{x-1} \ln u \, du, \int_1^e u^{x-1} (\ln u)^2 \, du, \dots, \int_1^e u^{x-1} (\ln u)^{m-\ell+1} \, du \right) \\
 & \rightarrow \sum_{\ell=1}^m \frac{k!}{(k-\ell)!} B_{m,\ell} \left(\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{m-\ell+2} \right), \quad x \rightarrow 0 \\
 & = \sum_{\ell=1}^m \frac{\binom{k}{\ell}}{\binom{m+\ell}{m}} \sum_{i=0}^{\ell} (-1)^i \binom{m+\ell}{i} S(m+\ell-i, \ell-i) \\
 & = \sum_{\ell=1}^m (-1)^{\ell} \frac{\binom{k}{\ell}}{\binom{m+\ell}{m}} \sum_{i=0}^{\ell} (-1)^i \binom{m+\ell}{m+i} S(m+i, i);
 \end{aligned} \tag{2.9}$$

(2) Similarly, when $m > k$, we have

$$\frac{d^m}{dx^m} \left[\left(\int_1^e u^{x-1} \, du \right)^k \right] \rightarrow \sum_{\ell=1}^k (-1)^{\ell} \frac{\binom{k}{\ell}}{\binom{m+\ell}{m}} \sum_{i=0}^{\ell} (-1)^i \binom{m+\ell}{m+i} S(m+i, i) \tag{2.10}$$

as $x \rightarrow 0$.

Since the convention that $\binom{k}{m} = 0$ for $m > k$, the equation (2.9) holds for all $m \geq 1$ and includes (2.10). Substituting (2.9) into (2.5) produces

$$S(n, k) = \binom{n}{k} \sum_{\ell=1}^{n-k} (-1)^{\ell} \frac{\binom{k}{\ell}}{\binom{n-k+\ell}{n-k}} \sum_{i=0}^{\ell} (-1)^i \binom{n-k+\ell}{n-k+i} S(n-k+i, i)$$

for $n-k \geq 1$. The formula (2.1) follows.

Interchanging two sums in (2.1) and then computing the inner sum result in

$$\begin{aligned}
 S(n, k) &= \binom{n}{k} \sum_{i=1}^{n-k} (-1)^i \left[\sum_{\ell=i}^{n-k} (-1)^{\ell} \frac{\binom{k}{\ell}}{\binom{n-k+\ell}{n-k}} \binom{n-k+\ell}{\ell-i} \right] S(n-k+i, i) \\
 &= \sum_{i=1}^{n-k} (-1)^i \binom{n}{k-i} \left[\sum_{\ell=i}^{n-k} (-1)^{\ell} \binom{k-i}{k-\ell} \right] S(n-k+i, i) \\
 &= \begin{cases} \sum_{i=1}^{n-k} (-1)^i \binom{n}{k-i} \left[\sum_{\ell=i}^{n-k} (-1)^{\ell} \binom{k-i}{k-\ell} \right] S(n-k+i, i), & k < n \leq 2k \\ \sum_{i=1}^k (-1)^i \binom{n}{k-i} \left[\sum_{\ell=i}^k (-1)^{\ell} \binom{k-i}{k-\ell} \right] S(n-k+i, i), & n > 2k \end{cases} \\
 &= \begin{cases} \sum_{i=2k-n}^{k-1} \binom{n}{i} \left[\sum_{\ell=0}^{i-(2k-n)} (-1)^{\ell} \binom{i}{\ell} \right] S(n-i, k-i), & k < n \leq 2k \\ \sum_{i=0}^{k-1} \binom{n}{i} \left[\sum_{\ell=0}^i (-1)^{\ell} \binom{i}{\ell} \right] S(n-i, k-i), & n > 2k \end{cases} \\
 &= \sum_{i=2k-n}^{k-1} \binom{n}{i} \left[\sum_{\ell=0}^{i-(2k-n)} (-1)^{\ell} \binom{i}{\ell} \right] S(n-i, k-i) \\
 &= (-1)^{n-2k} \sum_{i=2k-n}^{k-1} (-1)^i \binom{n}{i} \binom{i-1}{i-(2k-n)} S(n-i, k-i).
 \end{aligned}$$

The formula (2.2) follows. The proof of Theorem 2.1 is complete. \square

Remark 2.1. In [13, Theorem 1.1], a “diagonal” recurrence relation

$$s(n, k) = \sum_{m=1}^n \sum_{\ell=k-m}^{k-1} (-1)^{k+m-\ell} \binom{n}{\ell} \binom{\ell}{k-m} s(n-\ell, k-\ell) \quad (2.11)$$

for $n \geq k \geq 1$ was discovered for Stirling numbers of the first kind $s(n, k)$ which may be generated by

$$\frac{[\ln(1+x)]^k}{k!} = \sum_{n=k}^{\infty} s(n, k) \frac{x^n}{n!}, \quad |x| < 1. \quad (2.12)$$

As done in the derivation of (2.2), we may also interchange two sums in (2.1) and compute the inner sum as follows:

$$\begin{aligned} s(n, k) &= (-1)^k \sum_{\ell=k-n}^{k-1} (-1)^\ell \binom{n}{\ell} \left[\sum_{m=k-\ell}^n (-1)^m \binom{\ell}{k-m} \right] s(n-\ell, k-\ell) \\ &= (-1)^{n-k} \sum_{\ell=k-n}^{k-1} (-1)^\ell \binom{n}{\ell} \binom{\ell-1}{k-n-1} s(n-\ell, k-\ell), \quad n \geq k \geq 1, \end{aligned}$$

that is,

$$s(n, k) = (-1)^{n-k} \sum_{\ell=0}^{k-1} (-1)^\ell \binom{n}{\ell} \binom{\ell-1}{k-n-1} s(n-\ell, k-\ell), \quad n \geq k \geq 1, \quad (2.13)$$

where the conventions listed in (2.3) are also adopted. The recurrence relation (2.13) may also be called as a “diagonal” recurrence relation for Stirling numbers of the first kind $s(n, k)$.

The relations (2.11) and (2.13) are different from two “vertical” and two “horizontal” recurrence relations collected in [1, p. 215].

3. INEQUALITIES AND MONOTONICITY RELATED TO $S(n, k)$

After establishing and discussing “diagonal” recurrence relations for Stirling numbers of the first and second kinds $S(n, k)$ and $s(n, k)$, we now construct and deduce, with the help of the formula 2.5 and in light of properties of absolutely monotonic functions, some inequalities and monotonicity related to Stirling numbers of the second kind $S(n, k)$.

Theorem 3.1. *Let $m \geq 1$ be a positive integer and let $|a_{ij}|_m$ denote a determinant of order m with elements a_{ij} .*

- (1) *If a_i for $1 \leq i \leq m$ are non-negative integers, then*

$$\left| \frac{S(a_i + a_j + k, k)}{\binom{a_i + a_j + k}{k}} \right|_m \geq 0 \quad (3.1)$$

and

$$\left| (-1)^{a_i + a_j} \frac{S(a_i + a_j + k, k)}{\binom{a_i + a_j + k}{k}} \right|_m \geq 0 \quad (3.2)$$

hold true for all given $k \in \mathbb{N}$.

- (2) *Let $q = (q_1, q_2, \dots, q_n)$ be a real n -tuple of non-negative integers and let $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$ be non-increasing n -tuples of non-negative integers such that $a \succeq_q b$, that is,*

$$\sum_{i=1}^k q_i a_i \geq \sum_{i=1}^k q_i b_i$$

for $1 \leq k \leq n-1$ and

$$\sum_{i=1}^n q_i a_i = \sum_{i=1}^n q_i b_i.$$

Then the inequality

$$\prod_{i=1}^n \left[\frac{S(a_i + k, k)}{\binom{a_i + k}{k}} \right]^{q_i} \geq \prod_{i=1}^n \left[\frac{S(b_i + k, k)}{\binom{b_i + k}{k}} \right]^{q_i} \quad (3.3)$$

holds true for all given $k \in \mathbb{N}$.

Proof. A function f is said to be absolutely monotonic on an interval I if f has derivatives of all orders on I and $0 \leq f^{(n)}(x) < \infty$ for $x \in I$ and $n \geq 0$. See [2] and Chapter XIII in [10]. In [9] and [10, p. 367], it was recited that if f is an absolutely monotonic function on $[0, \infty)$, then

$$|f^{(a_i + a_j)}(x)|_m \geq 0 \quad (3.4)$$

and

$$|(-1)^{a_i + a_j} f^{(a_i + a_j)}(x)|_m \geq 0. \quad (3.5)$$

In [10, p. 368] and [11, p. 429], it was stated that if f is an absolutely monotonic function on $[0, \infty)$ and $a \succeq_q b$, then

$$\prod_{i=1}^n [f^{(a_i)}(x)]^{q_i} \geq \prod_{i=1}^n [f^{(b_i)}(x)]^{q_i}. \quad (3.6)$$

It is easy to see that

$$\left(\frac{e^x - 1}{x} \right)^{(m)} = \int_1^e u^{x-1} (\ln u)^m \, du, \quad m \geq 0,$$

see [16, 18] and plenty of closely-related references cited therein. This means that $\frac{e^x - 1}{x} = \int_1^e u^{x-1} \, du$ is absolutely monotonic on $(-\infty, \infty)$. As a result, the function

$$H_k(x) = \left(\frac{e^x - 1}{x} \right)^k = \left(\int_1^e u^{x-1} \, du \right)^k, \quad k \in \mathbb{N} \quad (3.7)$$

is also absolutely monotonic on $(-\infty, \infty)$ and, by the formula 2.5,

$$\lim_{x \rightarrow 0} H_k^{(\ell)}(x) = \frac{S(\ell + k, k)}{\binom{\ell + k}{k}}, \quad \ell \in \{0\} \cup \mathbb{N}. \quad (3.8)$$

Making use of inequalities (3.4), (3.5), and (3.6) and taking the limit $x \rightarrow 0$ find that

$$\begin{aligned} 0 &\leq |H_k^{(a_i + a_j)}(x)|_m \rightarrow \left| \frac{S(a_i + a_j + k, k)}{\binom{a_i + a_j + k}{k}} \right|_m, \\ 0 &\leq |(-1)^{a_i + a_j} H_k^{(a_i + a_j)}(x)|_m \rightarrow \left| (-1)^{a_i + a_j} \frac{S(a_i + a_j + k, k)}{\binom{a_i + a_j + k}{k}} \right|_m, \end{aligned}$$

and

$$\prod_{i=1}^n \left[\frac{S(a_i + k, k)}{\binom{a_i + k}{k}} \right]^{q_i} \leftarrow \prod_{i=1}^n [H_k^{(a_i)}(x)]^{q_i} \geq \prod_{i=1}^n [H_k^{(b_i)}(x)]^{q_i} \rightarrow \prod_{i=1}^n \left[\frac{S(b_i + k, k)}{\binom{b_i + k}{k}} \right]^{q_i}.$$

The proof of Theorem 3.1 is complete. \square

Corollary 3.1. *For any given $k \in \mathbb{N}$, the infinite sequence*

$$\left\{ \frac{S(n+k, k)}{\binom{n+k}{k}} \right\}_{n \geq 0} \quad (3.9)$$

is logarithmically convex with respect to n .

Proof. Letting

$$n = 2, \quad q_1 = q_2 = 1, \quad a_1 = \ell + 2, \quad a_2 = \ell, \quad \text{and} \quad b_1 = b_2 = \ell + 1$$

in the inequality (3.3) leads to

$$\frac{S(\ell+k+2, k)}{\binom{\ell+k+2}{k}} \frac{S(\ell+k, k)}{\binom{\ell+k}{k}} \geq \left[\frac{S(\ell+k+1, k)}{\binom{\ell+k+1}{k}} \right]^2, \quad \ell \geq 0. \quad (3.10)$$

As a result, the sequence (3.9) is logarithmically convex. The proof of Corollary 3.1 is complete. \square

Remark 3.1. The ideas employed in Theorem 3.1 and Corollary 3.1 have also been applied in the articles [14, 16, 19].

4. MONOTONICITY

After establishing diagonal recurrence relations and constructing inequalities related to Stirling numbers of the second kind $S(n, k)$, we are now in a position to create an infinite sequence in terms of Stirling numbers of the second kind $S(n, k)$ and to prove its increasing monotonicity.

Theorem 4.1. *For any fixed positive integers n, k with $n \geq k \geq 2$, let*

$$\mathfrak{S}_1(n, k) = S^2(n, k-1) - S(n, k-2)S(n, k). \quad (4.1)$$

Then the infinite sequence $\{\mathfrak{S}_1(n+m, k+m)\}_{m \geq 0}$ is strictly increasing with respect to m .

Proof. It is well known in combinatorics that Stirling numbers of the second kind $S(n, k)$ satisfy $S(0, 0) = 1$, $S(n, 0) = S(0, k) = 0$ for $n, k \geq 1$, and the “triangular” recurrence relation

$$S(n, k) = kS(n-1, k) + S(n-1, k-1) \quad (4.2)$$

for $n \geq k \geq 1$. See [1, p. 208]. Hence, the inequality (3.10) may be rearranged as

$$\begin{aligned} \frac{S(i+k+2, k)}{\binom{i+k+2}{k}} \frac{S(i+k, k)}{\binom{i+k}{k}} &= \frac{S(i+k+1, k-1) + kS(i+k+1, k)}{\binom{i+k+2}{k}} \frac{S(i+k, k)}{\binom{i+k}{k}} \\ &= \frac{S(i+k, k-2) + (2k-1)S(i+k, k-1) + k^2S(i+k, k)}{\binom{i+k+2}{k}} \frac{S(i+k, k)}{\binom{i+k}{k}} \\ &\geq \left[\frac{S(i+k+1, k)}{\binom{i+k+1}{k}} \right]^2 = \left[\frac{S(i+k, k-1) + kS(i+k, k)}{\binom{i+k+1}{k}} \right]^2. \end{aligned}$$

Replacing $i+k$ by n in the above inequality and simplifying give

$$\frac{S(n, k-2) + (2k-1)S(n, k-1) + k^2S(n, k)}{\binom{n+2}{k}} \frac{S(n, k)}{\binom{n}{k}} \geq \left[\frac{S(n, k-1) + kS(n, k)}{\binom{n+1}{k}} \right]^2,$$

that is, by the recurrence relation (4.2),

$$\begin{aligned}
 & \frac{S(n, k-2)S(n, k)}{\binom{n+2}{k}\binom{n}{k}} - \frac{S^2(n, k-1)}{\binom{n+1}{k}^2} \\
 & \geq \left[\frac{1}{\binom{n+1}{k}^2} - \frac{1}{\binom{n+2}{k}\binom{n}{k}} \right] k^2 S^2(n, k) + \left[\frac{2k}{\binom{n+1}{k}^2} - \frac{2k-1}{\binom{n+2}{k}\binom{n}{k}} \right] S(n, k-1)S(n, k), \\
 & \quad \frac{S(n, k-2)S(n, k) - S^2(n, k-1)}{\binom{n+2}{k}\binom{n}{k}} + \left[\frac{1}{\binom{n+2}{k}\binom{n}{k}} - \frac{1}{\binom{n+1}{k}^2} \right] S^2(n, k-1) \\
 & \geq \left[\frac{1}{\binom{n+1}{k}^2} - \frac{1}{\binom{n+2}{k}\binom{n}{k}} \right] k^2 S^2(n, k) + \left[\frac{2k}{\binom{n+1}{k}^2} - \frac{2k-1}{\binom{n+2}{k}\binom{n}{k}} \right] S(n, k-1)S(n, k), \\
 & \quad \frac{\mathfrak{S}_1(n, k)}{\binom{n+2}{k}\binom{n}{k}} \leq \left[\frac{2k-1}{\binom{n+2}{k}\binom{n}{k}} - \frac{2k}{\binom{n+1}{k}^2} \right] S(n, k-1)S(n, k) \\
 & \quad + \left[\frac{1}{\binom{n+2}{k}\binom{n}{k}} - \frac{1}{\binom{n+1}{k}^2} \right] [k^2 S^2(n, k) + S^2(n, k-1)], \\
 & \quad \mathfrak{S}_1(n, k) \leq \left[2k-1 - 2k \frac{\binom{n+2}{k}\binom{n}{k}}{\binom{n+1}{k}^2} \right] S(n, k-1)S(n, k) \\
 & \quad + \left[1 - \frac{\binom{n+2}{k}\binom{n}{k}}{\binom{n+1}{k}^2} \right] [k^2 S^2(n, k) + S^2(n, k-1)], \\
 & \quad \mathfrak{S}_1(n, k) \leq \left[1 - \frac{\binom{n+2}{k}\binom{n}{k}}{\binom{n+1}{k}^2} \right] [k^2 S^2(n, k) + 2kS(n, k-1)S(n, k) + S^2(n, k-1)] \\
 & \quad - S(n, k-1)S(n, k) \\
 & \quad = \left[1 - \frac{\binom{n+2}{k}\binom{n}{k}}{\binom{n+1}{k}^2} \right] [kS(n, k) + S(n, k-1)]^2 - S(n, k-1)S(n, k) \\
 & \quad = \frac{k}{(n+1)(n-k+2)} S^2(n+1, k) - S(n, k-1)S(n, k).
 \end{aligned}$$

In order to prove the increasing monotonicity of the infinite sequence $\{\mathfrak{S}_1(n+m, k+m)\}_{m \geq 0}$, it suffices to show

$$\frac{kS^2(n+1, k)}{(n+1)(n-k+2)} - S(n, k-1)S(n, k) \leq S^2(n+1, k) - S(n+1, k-1)S(n+1, k+1)$$

which may be reformulated as

$$\frac{S(n+1, k-1)S(n+1, k+1)}{S^2(n+1, k)} - \frac{S(n, k-1)S(n, k)}{S^2(n+1, k)} \leq \frac{(n+2)(n-k+1)}{(n+1)(n-k+2)}. \quad (4.3)$$

In [20, p. 698], it was proved by the recurrence relation in (4.2) and by induction that

$$(m-1)(n-m)S^2(n, m) > (m+1)(n-m+1)S(n, m-1)S(n, m+1) \quad (4.4)$$

for $2 \leq m \leq n-1$, which may be rewritten as

$$\frac{S(n+1, k-1)S(n+1, k+1)}{S^2(n+1, k)} < \frac{(k-1)(n-k+1)}{(k+1)(n-k+2)} \quad (4.5)$$

for $2 \leq k \leq n$. Therefore, in order to show (4.3), it is sufficient to verify

$$\frac{S(n, k-1)S(n, k)}{S^2(n+1, k)} \geq \frac{(k-1)(n-k+1)}{(k+1)(n-k+2)} - \frac{(n+2)(n-k+1)}{(n+1)(n-k+2)}$$

$$= -\frac{(n-k+1)(2n+k+3)}{(k+1)(n+1)(n-k+2)}$$

which is obvious. The proof of Theorem 4.1 is complete. \square

5. A CONJECTURE

Finally, motivated by Theorem 4.1, we pose the following conjecture.

Conjecture 5.1. For $k, \ell, n \in \mathbb{N}$, let $\mathfrak{S}_1(n, k)$ is defined by (4.1) and let

$$\mathfrak{S}_{\ell+1}(n, k) = \mathfrak{S}_\ell^2(n, k-1) - \mathfrak{S}_\ell(n, k-2)\mathfrak{S}_\ell(n, k) \quad (5.1)$$

and

$$\mathcal{S}_\ell(n, k) = \frac{\mathfrak{S}_{\ell+1}(n, k)}{\mathfrak{S}_\ell(n, k)} \quad (5.2)$$

for $n \geq k \geq \ell + 2$. Then the following claims are valid.

- (1) For fixed integers $\ell \in \mathbb{N}$ and $n \geq \ell + 3$, the finite sequence $\{\mathfrak{S}_\ell(n, k)\}_{\ell+1 \leq k \leq n}$ is logarithmically concave with respect to k .
- (2) For fixed integers $n \geq k \geq 3$, the finite sequence $\{\mathfrak{S}_\ell(n, k)\}_{1 \leq \ell \leq k-1}$ is strictly increasing with respect to ℓ .
- (3) For fixed integers $\ell \in \mathbb{N}$ and $n \geq k \geq \ell + 1$, the infinite sequence $\{\mathfrak{S}_\ell(n + m, k + m)\}_{m \geq 0}$ is strictly increasing with respect to m .
- (4) For fixed integers $\ell \in \mathbb{N}$ and $k \geq \ell + 1$, the infinite sequence $\{\mathfrak{S}_\ell(n, k)\}_{n \geq k}$ is strictly increasing with respect to n .
- (5) For fixed integers $n \geq k \geq \ell + 2$, the infinite sequence $\{\mathcal{S}_\ell(n + m, k + m)\}_{m \geq 0}$ is strictly increasing with respect to m .
- (6) For fixed integers $k \geq \ell + 2$, the infinite sequence $\{\mathcal{S}_\ell(n, k)\}_{n \geq k}$ is strictly increasing with respect to n .

Remark 5.1. There are some closely related results in the papers [3, 4, 6, 8, 15, 17] on Stirling numbers of the first and second kinds $s(n, k)$ and $S(n, k)$.

Remark 5.2. This paper is a revised version of the preprint [12].

REFERENCES

- [1] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Revised and Enlarged Edition, D. Reidel Publishing Co., Dordrecht and Boston, 1974.
- [2] B.-N. Guo and F. Qi, *A property of logarithmically absolutely monotonic functions and the logarithmically complete monotonicity of a power-exponential function*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **72** (2010), no. 2, 21–30.
- [3] B.-N. Guo and F. Qi, *Alternative proofs of a formula for Bernoulli numbers in terms of Stirling numbers*, Analysis (Berlin) **34** (2014), no. 2, 187–193; Available online at <http://dx.doi.org/10.1515/anly-2012-1238>.
- [4] B.-N. Guo and F. Qi, *An explicit formula for Bell numbers in terms of Stirling numbers and hypergeometric functions*, Glob. J. Math. Anal. **2** (2014), no. 4, 243–248; Available online at <http://dx.doi.org/10.14419/gjma.v2i4.3310>.
- [5] B.-N. Guo and F. Qi, *An explicit formula for Bernoulli numbers in terms of Stirling numbers of the second kind*, J. Anal. Number Theory **3** (2015), no. 1, 1–4; Available online at <http://dx.doi.org/10.12785/jant/????>.
- [6] B.-N. Guo and F. Qi, *Explicit formulae for computing Euler polynomials in terms of Stirling numbers of the second kind*, J. Comput. Appl. Math. **272** (2014), 251–257; Available online at <http://dx.doi.org/10.1016/j.cam.2014.05.018>.
- [7] B.-N. Guo and F. Qi, *Explicit formulas for special values of Bell polynomials of the second kind and Euler numbers*, ResearchGate Technical Report, available online at <http://dx.doi.org/10.13140/2.1.3794.8808>.
- [8] B.-N. Guo and F. Qi, *Some identities and an explicit formula for Bernoulli and Stirling numbers*, J. Comput. Appl. Math. **255** (2014), 568–579; Available online at <http://dx.doi.org/10.1016/j.cam.2013.06.020>.
- [9] D. S. Mitrinović and J. E. Pečarić, *On two-place completely monotonic functions*, Anzeiger Öster. Akad. Wiss. Math.-Naturwiss. Kl. **126** (1989), 85–88.

- [10] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993.
- [11] J. E. Pečarić, *Remarks on some inequalities of A. M. Fink*, J. Math. Anal. Appl. **104** (1984), no. 2, 428–431; Available online at [http://dx.doi.org/10.1016/0022-247X\(84\)90006-4](http://dx.doi.org/10.1016/0022-247X(84)90006-4).
- [12] F. Qi, *A recurrence formula, some inequalities, and monotonicity related to Stirling numbers of the second kind*, available online at <http://arxiv.org/abs/1402.2040>.
- [13] F. Qi, *A recurrence formula for the first kind Stirling numbers*, available online at <http://arxiv.org/abs/1310.5920>.
- [14] F. Qi, *An integral representation, complete monotonicity, and inequalities of Cauchy numbers of the second kind*, J. Number Theory **144** (2014), 244–255; Available online at <http://dx.doi.org/10.1016/j.jnt.2014.05.009>.
- [15] F. Qi, *Explicit formulas for computing Bernoulli numbers of the second kind and Stirling numbers of the first kind*, Filomat **28** (2014), no. 2, 319–327; Available online at <http://dx.doi.org/10.2298/FIL1402319Q>.
- [16] F. Qi, *Integral representations and properties of Stirling numbers of the first kind*, J. Number Theory **133** (2013), no. 7, 2307–2319; Available online at <http://dx.doi.org/10.1016/j.jnt.2012.12.015>.
- [17] F. Qi and B.-N. Guo, *Alternative proofs of a formula for Bernoulli numbers in terms of Stirling numbers*, Analysis (Berlin) **34** (2014), no. 3, 311–317; Available online at <http://dx.doi.org/10.1515/anly-2014-0003>.
- [18] F. Qi, Q.-M. Luo, and B.-N. Guo, *The function $(b^x - a^x)/x$: Ratio's properties*, In: *Analytic Number Theory, Approximation Theory, and Special Functions*, G. V. Milovanović and M. Th. Rassias (Eds), Springer, 2014, pp. 485–494; Available online at http://dx.doi.org/10.1007/978-1-4939-0258-3_16.
- [19] F. Qi and X.-J. Zhang, *An integral representation, some inequalities, and complete monotonicity of Bernoulli numbers of the second kind*; available online at <http://arxiv.org/abs/1301.6425>.
- [20] M. Sibuya, *Log-concavity of Stirling numbers and unimodality of Stirling distributions*, Ann. Inst. Statist. Math. **40** (1988), no. 4, 693–714; Available online at <http://dx.doi.org/10.1007/BF00049427>.
- [21] Z.-Z. Zhang and J.-Z. Yang, *Notes on some identities related to the partial Bell polynomials*, Tamsui Oxf. J. Inf. Math. Sci. **28** (2012), no. 1, 39–48.

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